An idea for determination of zeros of complex differential equations

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The research of complex differential equations regarding to zero solutions is very important, because zero function in a way symbolizes oscillation. Unlike Rolf Nevannlina's classical theory on evaluation of the number of zeros of the entire analytical function, and the entire solutions of complex differential equations, this paper suggests searching and finding of zero locations by simplified, generalized and accessible Sturm's theorems on the locations of zero solutions of oscillatory equations (but without theory of groups).

This paper is an introductory work into the whole problem of zero locations of complex differential equations. After the year 2000, the authors have noticed over 50 works about the evaluation of supremum of the number of zero solutions of complex differential equations, but they are of Nevannlina-type and do not contain any essential novelties.

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1. Introduction and preliminaries

If $w = f(z)$ is analytical function of complex variable z, the complex differential equation of oscillations is defined by

$$\frac{d^2w}{dz^2} + A(z)\, w = 0, \qquad (1)$$

where $A(z)$ is also analytical coefficient.

Equation (1) is not easily solved. Starting from 1926, Rolf Nevanlinna [1] developed a theory in which the number of zero solutions of this equation for certain $A(z)$, or in other words, the nature of solution oscillations $u(x, y)$ and $v(x, y)$ are estimated in very specific Nevanlinna-way, using the operations such as supremum of the number of zeros.This opens the way for development of entire functions, the bases of which have been set since Poincaré and Piccard.

In recent time, we listed more than 50 papers dealing only with zero solutions of an equation (1), which points to its actuality. At the same time, a group of mathematicians with D. Dimitrovski and M. Rajović at the head, has shown through [2], a completely elementary approach to the theory of oscillations.

The goal of our paper is to this simplified field of Sturm's zeros transfer from real domain to a complex domain. The novelty wouldn't be the estimation of Nevanlinna-type, but the zero location itself, and not only just for

$$w(z) = u(x,y) + iv(x,y), \tag{2}$$

but also separately only for real and imaginary parts. In this way we can also see better the amortization of oscillations with each other. Therefore, zeros of functions (2) are also important, separately $u(x,y) = 0$, $v(x,y) = 0$ together, as zeros $w(z)$.

We consequently have the extended problem of Rolf Nevannlina, with the goal of determination of the locations and counting the zeros in some narrower fields (since there are so many of them).

2. The basic elements and Sturm's zeros

Let's first mention some of the examples for the theorems which will come later.

Example 1. It is easy to show that the complex differential equations of the first order $\frac{dw}{dz} + A(z)w = 0$ have no zeros $w(z) = 0$, because solution is exponential function $w(z) = C\exp\left(-\int\limits_L A(z)\,dz\right)$, $C = \alpha + i\beta$, $A(z) = a(x,y) + ib(x,y)$, L – opened curve.

We see that only solutions $\mathrm{Re}w(z)$ and $\mathrm{Im}w(z)$ can have their zeros, which are countless, but the point is that there are no common zeros.

Example 2. Let's take now the simplest equation (1) with complex coefficients $A(z) = (\alpha + i\beta)$. Here will be applicable a classic procedure of finding the particular integral where we have the characteristic equation $r^2 + (\alpha + i\beta) = 0$ and solutions $r_{1/2} = \pm\sqrt{-(\alpha + i\beta)}$. The general solution is $w(z) = c_1e^{r_1z} + c_2e^{r_2z}$. Is the solution oscillatory?

Let's take the simplest case

$$\frac{d^2w}{dz^2} + w = 0. \tag{3}$$

According to above mentioned, we easily obtain $r_{1/2} = \pm i$, which independent solutions are $w_1 = e^{-y}(\cos x + i\sin x)$, $w_2 = e^{y}(\cos x - i\sin x)$. Therefore, the general solution is $w(z) = (c_1e^{-y} + c_2e^{y})\cos x + i(c_1e^{-y} - c_2e^{y})\sin x$. A question arises whether at the same time applies $u(x,y) = 0$ and $v(x,y) = 0$?

As for positive y the expressions in brackets are not equal zero, regardless the fact that c_1 and c_2 can be complex it follows that at the same time should be $\cos x = 0$ and $\sin x = 0$.

However, that cannot be at the same time, because Sturm's zeros of harmonic oscillations $x^I = (2k-1)\frac{\pi}{2}$, $k = 1, 2, \ldots$ and $x^{II} = n\pi$, $n = 0, 1, 2, \ldots$, are completely separeted from each other and they are different, so they will never coincide. It implies that equation (3) has solutions which do not have zeros, since $u(x,y) = 0$ and $v(x,y) = 0$ do not have common zeros. Functions itself $\mathrm{Re}w_1(z)$, $\mathrm{Im}w_1(z)$ and $\mathrm{Re}w_2(z)$, $\mathrm{Im}w_2(z)$ can each have their own zeros, and these are the oscillatory functions of two real variables.

Example 3. Let us again consider the complex differential equation (3) which cannot be solved elementary. Using elementary arithmetical operations in the set of complex numbers, when in (3) we separate real and imaginary parts, we get the system of two real partial equations

$$\frac{\partial^2 u}{\partial x^2} = -xu(x,y) + yv(x,y),$$
$$\frac{\partial^2 v}{\partial x^2} = -xv(x,y) - yu(x,y).$$

This system can be transformed into Sturm's ordinary differential equations by each ray, $y = kx$, $0 \leq k < \infty$. For example, if $k = 0$, i. e. on Ox-axis we obtain

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + xu(x) = 0, \\ &\frac{\partial^2 v}{\partial x^2} + xv(x) = 0. \end{aligned} \tag{4}$$

We can see that this is the same ordinary differential equation (connected to the Riccati equation in theory, and it can be solved by quadratures) and it is connected to the Bessel equation (see [3], page 40). We are going to solve the equation (4) generally, as an oscillating equation $y'' + a(x)y = 0$, for $a(x) > 0$ (see [2]), by our iteration sequence method. In our case, its two linearly independent solutions for both equations in (4), for u (the same is for v) are

$$\begin{aligned} u_1 &= 1 - \iint xdx^2 + \iint xdx^2 \iint xdx^2 - \iint xdx^2 \iint xdx^2 \iint xdx^2 + \ldots, \\ u_2 &= x - \iint x^2dx^2 + \iint xdx^2 \iint x^2dx^2 - \iint xdx^2 \iint xdx^2 \iint x^2dx^2 + \ldots \end{aligned} \tag{5}$$

We proved that these are oscillating functions and for them we introduced signs $u_1 = \cos_{a(x)} x = \cos_x x$, $u_2 = \sin_{a(x)} x = \sin_x x$.

The zeros of these oscillations are Sturm's zeros approximately located in the solutions of equations: for sine solution: $x\sqrt{x} = n\pi$, $n = 0, 1, 2, \ldots$, and for cosine solution: $x\sqrt{x} = (2k-1)\frac{\pi}{2}$, $k = 1, 2, \ldots$ These zeros never coincide, because they are Sturm's, separeted, simple zeros and they are both for u or v. It cannot be $u \equiv v$, because then $w(z) \equiv u + iv = (1+i)u$ and this is not an analytical function so Cauchy-Riemman conditions are not applicable. It follows that if $u_1 = \cos_x x$ is one solution of (5), then $v_1 = \sin_x x$ is the same solution, and vice versa, but it is always $u_1 = v_2$, as well as $u_2 = v_1$. Therefore, solutions of complex differential equation (3), on Ox-axis are: $w_1(z) \equiv u_1 + iv_1 = \cos_x x + i\sin_x x$, $w_2(z) \equiv u_2 + iv_2 = \sin_x x + i\cos_x x$ and they have not zeros. We would act similarly on each ray $y = kx$, $0 \leq k < \infty$. This is not valid for Oy-axis because k is indefinite.

Therefore, it follows

Theorem 1. *The complex differential equation (3) with coefficient which is the simple linear function of z, has solution w which has no zeros on Ox-axis.*

We could similarly apply for complex equations $\frac{d^2w}{dz^2} + P_n(z)w = 0$, where $P_n(z)$ is polynomial of n-th degree in respect of independent variables z and with arbitrary complex coefficients.

We can also here formulate analogous theorems about zeros. Furthemore, if there are zeros, we can, at the same time, provide their locations, first only for u, then only for v and then for $w(z) = u + iv$.

Example 4. Let us consider complex differential equation

$$\frac{d^2w}{dz^2}+e^z w=0. \tag{6}$$

In the same way as in the previous example, when we separate the real part from the imaginary one, we get the system of partial equations of the second order

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=e^x\left(v\left(x,y\right)\sin y-u\left(x,y\right)\cos y\right),\\ \frac{\partial^2 v}{\partial x^2}&=-e^x\left(u\left(x,y\right)\sin y+v\left(x,y\right)\cos y\right).\end{aligned} \tag{7}$$

We can turn (7) into Sturm's ordinary differential equations by each ray $y=kx$, $0\leq k<\infty$, but for the sake of shortness we will do this again for $k=0$. Hence, only on Ox-axis, we will have ordinary differential equations

$$\begin{aligned}\frac{d^2u}{dx^2}+e^x u\left(x\right)=0,\\ \frac{d^2v}{dx^2}+e^x v\left(x\right)=0.\end{aligned}$$

Therefore, again we got the same equations for u and for v. The solutions of the equation $y''+e^x y=0$, for $a\left(x\right)=e^x>0$, according to [2], are

$$\begin{aligned}u_1&=1-\iint e^x dx^2+\iint e^x dx^2\iint e^x dx^2-\iint e^x dx^2\iint e^x dx^2\iint e^x dx^2+\ldots,\\ u_2&=x-\iint xe^x dx^2+\iint x dx^2\iint xe^x dx^2-\iint x dx^2\iint x dx^2\iint xe^x dx^2+\ldots\end{aligned}$$

The same is also applies for v. These are oscillating functions, $u_1=\cos_{e^x}x$, $u_2=\sin_{e^x}x$, and the similar $u_1=v_2$ and $u_2=v_1$ will apply. Sturm's zeros will be found by using frequency function $F\left(x\right)=x\sqrt{a\left(x\right)}=x\sqrt{e^x}=xe^{x/2}$, and according to our theorem they are: for sine solution in the roots of equation $x\sqrt{e^x}=n\pi$, $n=0,1,2,\ldots$, and for cosine solution in the roots of equation $x\sqrt{e^x}=\left(2k-1\right)\frac{\pi}{2}$, $k=1,2,\ldots$ Also according to Sturm's theorems the zeros of u_1 are extremes of v_2, and vice versa extremes of u_1 are zeros of v_2.

Geometrically, zero solutions are cross sections of curve $y=F\left(x\right)=x\sqrt{e^x}$ with horizontals $y=n\pi$, $n=0,1,2,\ldots$, and $y=\left(2k-1\right)\frac{\pi}{2}$, $k=1,2,\ldots$

We can now formulate the theorem.

Theorem 2. *Complex differential equation (6) has no zeros on real Ox-axis, for $w=u$ or $w=v$, but each part $u=u\left(x,y\right)$ and $v=v\left(x,y\right)$ has countless zeros, respectively.*

However, they are not congruent, so it is always on Ox-axis, $w\neq 0$. Also, this doesn't mean that this will remain on each ray $y=kx$, because then the system (7) is

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=e^x\left(v\left(x,kx\right)\sin kx-u\left(x,kx\right)\cos kx\right),\\ \frac{\partial^2 v}{\partial x^2}&=-e^x\left(u\left(x,kx\right)\sin kx+v\left(x,kx\right)\cos kx\right).\end{aligned}$$

As the right side of these equations now is a function from k and from x, through e^x, $\cos kx$, $\sin kx$ the analysis is more complex.

Basically, this gives the possibility of finding zero locations according to some definite field, primarily on rays $y = kx$ and then by arbitrary smooth curve $y = g(x)$. In this way we can also approach the classical theory of Rolfo Nevanlinna (see [1]), supplemented also with zero locations, because it doesn't give them but gives only evaluation of the supremum of the number of zeros in the final field.

References

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